Chapter 22

Additional Topics in Math

In addition to the questions in Heart of Algebra, Problem Solving and Data Analysis, and Passport to Advanced Math, the SAT Math Test includes several questions that are drawn from areas of geometry, trigonometry, and the arithmetic of complex numbers. They include both multiple-choice and student-produced response questions. On some questions, the use of a calculator is not permitted; on others, the use of a calculator is allowed.

Let's explore the content and skills assessed by these questions.

REMEMBER

6 of the 58 questions (approximately 10%) on the SAT Math Test will be drawn from Additional Topics in Math, which includes geometry, trigonometry, and the arithmetic of complex numbers.

Geometry

The SAT Math Test includes questions that assess your understanding of the key concepts in the geometry of lines, angles, triangles, circles, and other geometric objects. Other questions may also ask you to find the area, surface area, or volume of an abstract figure or a real-life object. You do not need to memorize a large collection of formulas. Many of the geometry formulas are provided in the Reference Information at the beginning of each section of the SAT Math Test, and less commonly used formulas required to answer a question are given with the question.

REMEMBER

You do not need to memorize a large collection of geometry formulas. Many geometry formulas are provided on the SAT Math Test in the Reference section of the directions.

To answer geometry questions on the SAT Math Test, you should recall the geometry definitions learned prior to high school and know the essential concepts extended while learning geometry in high school. You should also be familiar with basic geometric notation.

Here are some of the areas that may be the focus of some questions on the SAT Math Test.

- Lines and angles
  - Lengths and midpoints
  - Vertical angles
  - Straight angles and the sum of the angles about a point

- — Properties of parallel lines and the angles formed when parallel lines are cut by a transversal
- — Properties of perpendicular lines

▶ Triangles and other polygons

- — Right triangles and the Pythagorean theorem
- — Properties of equilateral and isosceles triangles
- — Properties of 30°-60°-90° triangles and 45°-45°-90° triangles
- — Congruent triangles and other congruent figures
- — Similar triangles and other similar figures
- — The triangle inequality
- — Squares, rectangles, parallelograms, trapezoids, and other quadrilaterals
- — Regular polygons

▶ Circles

- — Radius, diameter, and circumference
- — Measure of central angles and inscribed angles
- — Arc length and area of sectors
- — Tangents and chords

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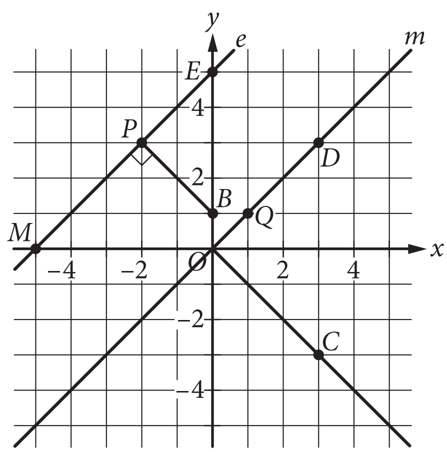
The triangle inequality theorem states that for any triangle, the length of any side of the triangle must be less than the sum of the other two sides of the triangle and greater than the difference of the other two sides.

You should be familiar with the geometric notation for points and lines, line segments, angles and their measures, and lengths.



In the figure above, the xy-plane has origin O. The values of x on the horizontal x-axis increase as you move to the right, and the values of y on the vertical y-axis increase as you move up. Line e contains point P, which has coordinates (−2, 3), and point E, which has coordinates (0, 5). Line m passes through the origin O (0, 0) and the point Q (1, 1).

Lines e and m are parallel — they never meet. This is written $e \parallel m$.

You will also need to know the following notation:

- $\overleftrightarrow{PE}$: the line containing the points P and E (this is the same as line e)
- $\overline{PE}$ or segment PE: the line segment with endpoints P and E
- PE: the length of segment PE (you can write $PE = 2\sqrt{2}$)
- $\overrightarrow{PE}$: the ray starting at point P and extending indefinitely in the direction of E
- $\overrightarrow{EP}$: the ray starting at point E and extending indefinitely in the direction of P
- $\angle DOC$: the angle formed by $\overrightarrow{OD}$ and $\overrightarrow{OC}$
- $m\angle DOC$: the measure of $\angle DOC$ (you can write $m\angle DOC = 90°$)
- $\triangle PEB$: the triangle with vertices P, E, and B
- $BPMO$: the quadrilateral with vertices B, P, M, and O
- $\overline{BP} \perp \overline{PM}$: segment BP is perpendicular to segment PM (you should also recognize that the small square within $\angle BPM$ means this angle is a right angle)

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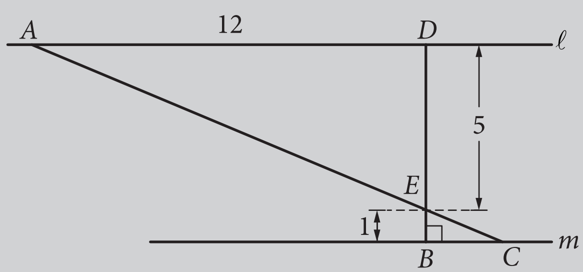
Familiarize yourself with these notations in order to avoid confusion on test day.

EXAMPLE 1



In the figure above, line ℓ is parallel to line m, segment BD is perpendicular to line m, and segment AC and segment BD intersect at E. What is the length of segment AC?

Since segment AC and segment BD intersect at E, $\angle AED$ and $\angle CEB$ are vertical angles, and so the measure of $\angle AED$ is equal to the measure of $\angle CEB$. Since line ℓ is parallel to line m, $\angle BCE$ and $\angle DAE$ are alternate interior angles of parallel lines cut by a transversal, and so the measure of $\angle BCE$ is equal to the measure of $\angle DAE$. By the angle-angle theorem, $\triangle AED$ is similar to $\triangle CEB$, with vertices A, E, and D corresponding to vertices C, E, and B, respectively.

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A shortcut here is remembering that 5, 12, 13 is a Pythagorean triple (5 and 12 are the lengths of the sides of the right triangle, and 13 is the length of the hypotenuse). Another common Pythagorean triple is 3, 4, 5.

Also, $\triangle AED$ is a right triangle, so by the Pythagorean theorem, $AE = \sqrt{AD^2 + DE^2} = \sqrt{12^2 + 5^2} = \sqrt{169} = 13$. Since ΔAED is similar to ΔCEB, the ratios of the lengths of corresponding sides of the two triangles are in the same proportion, which is $\frac{ED}{EB} = \frac{5}{1} = 5$. Thus, $\frac{AE}{EC} = \frac{13}{EC} = 5$, and so $EC = \frac{13}{5}$. Therefore, $AC = AE + EC = 13 + \frac{13}{5} = \frac{78}{5}$.

Note some of the key concepts that were used in Example 1:

- Vertical angles have the same measure.
- When parallel lines are cut by a transversal, the alternate interior angles have the same measure.
- If two angles of a triangle are congruent to (have the same measure as) two angles of another triangle, the two triangles are similar.
- The Pythagorean theorem.
- If two triangles are similar, then all ratios of lengths of corresponding sides are equal.
- If point E lies on line segment AC, then $AC = AE + EC$.

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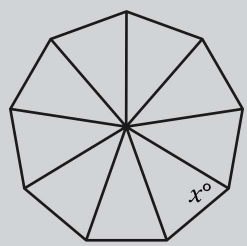
Note how Example 1 requires the knowledge and application of numerous fundamental geometry concepts. Develop mastery of the fundamental concepts and practice applying them on test-like questions.

Note that if two triangles or other polygons are similar or congruent, the order in which the vertices are named does *not* necessarily indicate how the vertices correspond in the similarity or congruence. Thus, it was stated explicitly in Example 1 that "$\triangle AED$ is similar to $\triangle CEB$, with vertices A, E, and D corresponding to vertices C, E, and B, respectively."

EXAMPLE 2



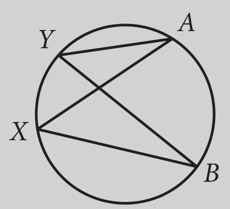
In the figure above, a regular polygon with 9 sides has been divided into 9 congruent isosceles triangles by line segments drawn from the center of the polygon to its vertices. What is the value of x?

The sum of the measures of the angles around a point is 360°. Since the 9 triangles are congruent, the measures of each of the 9 angles are the same. Thus, the measure of each angle is $\frac{360°}{9} = 40°$. In any triangle, the sum of the measures of the interior angles is 180°. So in each triangle, the sum of the measures of the remaining two angles is 180° – 40° = 140°. Since each triangle is isosceles, the measure of each of these two angles is the same. Therefore, the measure of each of these angles is $\frac{140°}{2} = 70°$. Hence, the value of x is 70.

Note some of the key concepts that were used in Example 2:

- The sum of the measures of the angles about a point is 360°.
- Corresponding angles of congruent triangles have the same measure.
- The sum of the measure of the interior angles of any triangle is 180°.
- In an isosceles triangle, the angles opposite the sides of equal length are of equal measure.

EXAMPLE 3



In the figure above, $\angle AXB$ and $\angle AYB$ are inscribed in the circle. Which of the following statements is true?

A) The measure of $\angle AXB$ is greater than the measure of $\angle AYB$.
B) The measure of $\angle AXB$ is less than the measure of $\angle AYB$.
C) The measure of $\angle AXB$ is equal to the measure of $\angle AYB$.
D) There is not enough information to determine the relationship between the measure of $\angle AXB$ and the measure of $\angle AYB$.

Choice C is correct. Let the measure of arc $\overset{\frown}{AB}$ be $d°$. Since $\angle AXB$ is inscribed in the circle and intercepts arc $\overset{\frown}{AB}$, the measure of $\angle AXB$ is equal to half the measure of arc $\overset{\frown}{AB}$. Thus, the measure of $\angle AXB$ is $\frac{d°}{2}$. Similarly, since $\angle AYB$ is also inscribed in the circle and intercepts arc $\overset{\frown}{AB}$, the measure of $\angle AYB$ is also $\frac{d°}{2}$. Therefore, the measure of $\angle AXB$ is equal to the measure of $\angle AYB$.

Note the key concept that was used in Example 3:

- The measure of an angle inscribed in a circle is equal to half the measure of its intercepted arc.

You also should know this related concept:

- The measure of a central angle in a circle is equal to the measure of its intercepted arc.

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At first glance, it may appear as though there is not enough information to determine the relationship between the two angle measures. One key to this question is identifying what is the same about the two angle measures. In this case, both angles intercept arc $\overset{\frown}{AB}$.

You should also be familiar with notation for arcs and circles on the SAT:

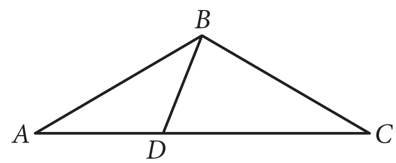
- A circle may be named by the point at its center. So, the center of a circle M would be point M.
- An arc named with only its two endpoints, such as $\overset{\frown}{AB}$, will always refer to a minor arc. A minor arc has a measure that is less than 180°.
- An arc may also be named with three points: the two endpoints and a third point that the arc passes through. So, $\overset{\frown}{ACB}$ has endpoints at A and B and passes through point C. Three points may be used to name a minor arc or an arc that has a measure of 180° or more.

REMEMBER

Figures are drawn to scale on the SAT Math Test unless explicitly stated otherwise. If a question states that a figure is not drawn to scale, be careful not to make unwarranted assumptions about the figure.

In general, figures that accompany questions on the SAT Math Test are intended to provide information that is useful in answering the question. They are drawn as accurately as possible EXCEPT in a particular question when it is stated that the figure is not drawn to scale. In general, even in figures not drawn to scale, the relative positions of points and angles may be assumed to be in the order shown. Also, line segments that extend through points and appear to lie on the same line may be assumed to be on the same line. A point that appears to lie on a line or curve may be assumed to lie on the line or curve.

The text "Note: Figure not drawn to scale." is included with the figure when degree measures may not be accurately shown and specific lengths may not be drawn proportionally. The following example illustrates what information can and cannot be assumed from a figure not drawn to scale.



Note: Figure not drawn to scale.

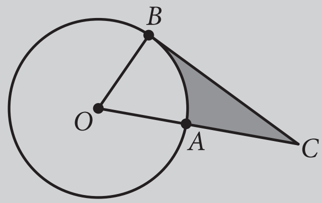
A question may refer to a triangle such as ABC above. Although the note indicates that the figure is not drawn to scale, you may assume the following from the figure:

- ABD and DBC are triangles.
- D is between A and C.
- A, D, and C are points on a line.
- The length of $\overline{AD}$ is less than the length of $\overline{AC}$.
- The measure of angle ABD is less than the measure of angle ABC.

You may *not* assume the following from the figure:

- The length of $\overline{AD}$ is less than the length of $\overline{DC}$.
- The measures of angles BAD and DBA are equal.
- The measure of angle DBC is greater than the measure of angle ABD.
- Angle DBC is a right angle.

EXAMPLE 4



In the figure above, O is the center of the circle, segment BC is tangent to the circle at B, and A lies on segment OC. If $OB = AC = 6$, what is the area of the shaded region?

A) $18\sqrt{3} - 3\pi$
B) $18\sqrt{3} - 6\pi$
C) $36\sqrt{3} - 3\pi$
D) $36\sqrt{3} - 6\pi$

Since segment BC is tangent to the circle at B, it follows that $\overline{BC} \perp \overline{OB}$, and so triangle OBC is a right triangle with its right angle at B. Since $OB = 6$ and OB and OA are both radii of the circle, $OA = OB = 6$, and $OC = OA + AC = 12$. Thus, triangle OBC is a right triangle with the length of the hypotenuse ($OC = 12$) twice the length of one of its legs ($OB = 6$). It follows that triangle OBC is a 30°-60°-90° triangle with its 30° angle at C and its 60° angle at O. The area of the shaded region is the area of triangle OBC minus the area of the sector bounded by radii OA and OB.

In the 30°-60°-90° triangle OBC, the length of side OB, which is opposite the 30° angle, is 6. Thus, the length of side BC, which is opposite the 60° angle, is $6\sqrt{3}$. Hence, the area of triangle OBC is $\frac{1}{2}(6)(6\sqrt{3}) = 18\sqrt{3}$. Since the sector bounded by radii OA and OB has central angle 60°, the area of this sector is $\frac{60}{360} = \frac{1}{6}$ of the area of the circle. Since the circle has radius 6, its area is $\pi(6)^2 = 36\pi$, and so the area of the sector is $\frac{1}{6}(36\pi) = 6\pi$. Therefore, the area of the shaded region is $18\sqrt{3} - 6\pi$, which is choice B.

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On complex multistep questions such as Example 4, start by identifying the task (finding the area of the shaded region) and considering the intermediate steps that you'll need to solve for (the area of triangle *OBC* and the area of sector *OBA*) in order to get to the final answer. Breaking up this question into a series of smaller questions will make it more manageable.

Note some of the key concepts that were used in Example 4:

- A tangent to a circle is perpendicular to the radius of the circle drawn to the point of tangency.
- Properties of 30°-60°-90° triangles.
- Area of a circle.
- The area of a sector with central angle $x°$ is equal to $\frac{x}{360}$, the area of the entire circle.

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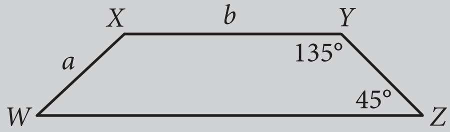
Arc length, area of a sector, and central angle are all proportional to each other in a circle. This proportionality is written as:

arc length / circumference =
central angle / 360 degrees =
area of a sector / area of a circle

EXAMPLE 5



Trapezoid $WXYZ$ is shown above. How much greater is the area of this trapezoid than the area of a parallelogram with side lengths a and b and base angles of measure 45° and 135°?

A) $\frac{1}{2}a^2$

B) $\sqrt{2}a^2$

C) $\frac{1}{2}ab$

D) $\sqrt{2}ab$

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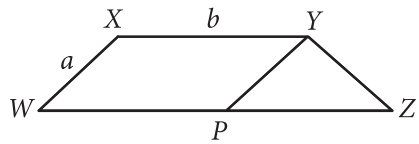
Note how drawing the parallelogram within trapezoid *WXYZ* makes it much easier to compare the areas of the two shapes, minimizing the amount of calculation needed to arrive at the solution. Be on the lookout for time-saving shortcuts such as this one.

In the figure, draw a line segment from Y to the point P on side WZ of the trapezoid such that $\angle YPW$ has measure 135°, as shown in the figure below.



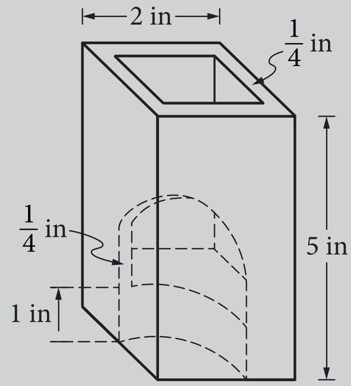
Since in trapezoid $WXYZ$ side XY is parallel to side WZ, it follows that $WXYP$ is a parallelogram with side lengths a and b and base angles of measure 45° and 135°. Thus, the area of the trapezoid is greater than a parallelogram with side lengths a and b and base angles of measure 45° and 135° by the area of triangle PYZ. Since $\angle YPW$ has measure 135°, it follows that $\angle YPZ$ has measure 45°. Hence, triangle PYZ is a 45°-45°-90° triangle with legs of length a. Therefore, its area is $\frac{1}{2}a^2$, which is choice A.

Note some of the key concepts that were used in Example 5:

- Properties of trapezoids and parallelograms
- Area of a 45°-45°-90° triangle

Some questions on the SAT Math Test may ask you to find the area, surface area, or volume of an object, possibly in a real-life context.

EXAMPLE 6



Note: Figure not drawn to scale.

A glass vase is in the shape of a rectangular prism with a square base. The figure above shows the vase with a portion cut out. The external dimensions of the vase are height 5 inches (in), with a square base of side length 2 inches. The vase has a solid base of height 1 inch, and the sides are each $\frac{1}{4}$ inch thick. Which of the following is the volume, in cubic inches, of the glass used in the vase?

A) 6
B) 8
C) 9
D) 11

The volume of the glass used in the vase can be calculated by subtracting the inside volume of the vase from the outside volume of the vase. Both the inside and outside volumes are from different-sized rectangular prisms. The outside dimensions of the prism are 5 inches by 2 inches by 2 inches, so its volume, including the glass, is $5 \times 2 \times 2 = 20$ cubic inches. For the inside volume of the vase, since it has a solid base of height 1 inch, the height of the prism removed is $5 - 1 = 4$ inches. In addition, each side of the vase is $\frac{1}{4}$ inch thick, so each side length of the inside volume is $2 - \frac{1}{4} - \frac{1}{4} = \frac{3}{2}$ inches. Thus, the inside volume of the vase removed is $4 \times \frac{3}{2} \times \frac{3}{2} = 9$ cubic inches. Therefore, the volume of the glass used in the vase is $20 - 9 = 11$ cubic inches, which is choice D.

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Pay close attention to detail on a question such as Example 6. You must take into account the fact that the vase has a solid base of height 1 inch when subtracting the inside volume of the vase from the outside volume of the vase.

Coordinate Geometry

Questions on the SAT Math Test may ask you to use the coordinate plane and equations of lines and circles to describe figures. You may be asked to create the equation of a circle given the figure or use the structure of a given equation to determine a property of a figure in the coordinate plane. You

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You should know that the graph of $(x - a)^2 + (y - b)^2 = r^2$ in the xy-plane is a circle with center (a, b) and radius r.

should know that the graph of $(x - a)^2 + (y - b)^2 = r^2$ in the xy-plane is a circle with center (a, b) and radius r.

EXAMPLE 7

$$x^2 + (y + 1)^2 = 4$$

The graph of the equation above in the xy-plane is a circle. If the center of this circle is translated 1 unit up and the radius is increased by 1, which of the following is an equation of the resulting circle?

A) $x^2 + y^2 = 5$
B) $x^2 + y^2 = 9$
C) $x^2 + (y + 2)^2 = 5$
D) $x^2 + (y + 2)^2 = 9$

The graph of the equation $x^2 + (y + 1)^2 = 4$ in the xy-plane is a circle with center $(0, -1)$ and radius $\sqrt{4} = 2$. If the center is translated 1 unit up, the center of the new circle will be $(0, 0)$. If the radius is increased by 1, the radius of the new circle will be 3. Therefore, an equation of the new circle in the xy-plane is $x^2 + y^2 = 3^2 = 9$, so choice B is correct.

EXAMPLE 8

$$x^2 + 8x + y^2 - 6y = 24$$

The graph of the equation above in the xy-plane is a circle. What is the radius of the circle?

The given equation is not in the standard form $(x - a)^2 + (y - b)^2 = r^2$. You can put it in standard form by completing the square. Since the coefficient of x is 8 and the coefficient of y is -6, you can write the equation in terms of $(x + 4)^2$ and $(y - 3)^2$ as follows:

$$x^2 + 8x + y^2 - 6y = 24$$

$$(x^2 + 8x + 16) - 16 + (y^2 - 6y + 9) - 9 = 24$$

$$(x + 4)^2 - 16 + (y - 3)^2 - 9 = 24$$

$$(x + 4)^2 + (y - 3)^2 = 24 + 16 + 9 = 49 = 7^2$$

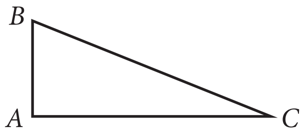
Therefore, the radius of the circle is 7. (Also, the center of the circle is $(-4, 3)$.)

Trigonometry and Radians

Questions on the SAT Math Test may ask you to apply the definitions of right triangle trigonometry. You should also know the definition of radian measure; you may also need to convert between angle measure in degrees and radians.

You may need to evaluate trigonometric functions at benchmark angle measures such as 0, $\frac{\pi}{6}$, $\frac{\pi}{4}$, $\frac{\pi}{3}$, and $\frac{\pi}{2}$ radians (which are equal to the angle measures 0°, 30°, 45°, 60°, and 90°, respectively). You will *not* be asked for values of trigonometric functions that require a calculator.

For an acute angle, the trigonometric functions sine, cosine, and tangent can be defined using right triangles. (Note that the functions are often abbreviated as sin, cos, and tan, respectively.)



For $\angle C$ in the right triangle above:

- $\sin(\angle C) = \frac{AB}{BC} = \frac{\text{length of leg opposite } \angle C}{\text{length of hypotenuse}}$
- $\cos(\angle C) = \frac{AC}{BC} = \frac{\text{length of leg adjacent to } \angle C}{\text{length of hypotenuse}}$
- $\tan(\angle C) = \frac{AB}{AC} = \frac{\text{length of leg opposite } \angle C}{\text{length of leg adjacent to } \angle C} = \frac{\sin(\angle C)}{\cos(\angle C)}$,

The functions will often be written as sin C, cos C, and tan C, respectively.

Note that the trigonometric functions are actually functions of the *measures* of an angle, not the angle itself. Thus, if the measure of $\angle C$ is, say, 30°, you can write sin(30°), cos(30°), and tan(30°), respectively.

Also note that $\sin B = \frac{\text{length of leg opposite } \angle B}{\text{length of hypotenuse}} = \frac{AC}{BC} = \cos C$. This is the complementary angle relationship: $\sin(x°) = \cos(90° - x°)$.

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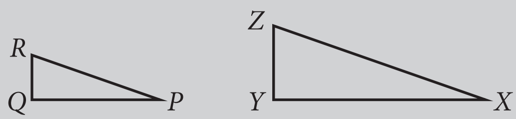
The acronym "SOHCAHTOA" may help you remember how to compute sine, cosine, and tangent. SOH stands for Sine equals Opposite over Hypotenuse, CAH stands for Cosine equals Adjacent over Hypotenuse, and TOA stands for Tangent equals Opposite over Adjacent.

EXAMPLE 9

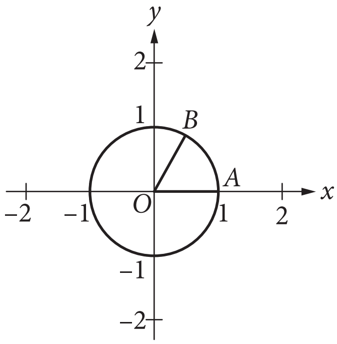


In the figure above, right triangle PQR is similar to right triangle XYZ, with vertices P, Q, and R corresponding to vertices X, Y, and Z, respectively. If $\cos R = 0.263$ what is the value of $\cos Z$?

By the definition of cosine, $\cos R = \frac{RQ}{RP}$ and $\cos Z = \frac{ZY}{ZX}$. Since triangle PQR is similar to triangle XYZ, with vertices P, Q, and R corresponding to vertices X, Y, and Z, respectively, the ratios $\frac{RQ}{RP}$ and $\frac{ZY}{ZX}$ are equal. Therefore, since $\cos R = \frac{RQ}{RP} = 0.263$, it follows that $\cos Z = \frac{ZY}{ZX} = 0.263$.

Note that this is why, to find the values of the trigonometric functions of, say, $d°$, you can use *any* right triangle with an acute angle of measure $d°$ and then take the appropriate ratio of lengths of sides.

Note that since an acute angle of a right triangle has measure between 0° and 90°, exclusive, right triangles can be used only to find values of trigonometric functions for angles with measures between 0° and 90°, exclusive. The definitions of sine, cosine, and tangent can be extended to all values. This is done using radian measure and the unit circle.



The circle above has radius 1 and is centered at the origin, O. An angle in the coordinate plane is said to be in standard position if it meets these two conditions: (1) its vertex lies at the origin and (2) one of its sides lies along the positive x-axis. Since angle AOB above, formed by segments OA and OB, meets both these conditions, it is said to be in *standard position*. As segment OB, also called the *terminal side* of angle AOB, rotates counterclockwise about the circle, while OA is anchored along the x-axis, the *radian* measure of angle AOB is defined to be the length s of the arc that angle AOB intercepts on the unit circle. In turn, $m\angle AOB$ is s radians.

When an acute angle AOB is in standard position within the unit circle, the x-coordinate of point B is $\cos(\angle AOB)$, and the y-coordinate of point B is $\sin(\angle AOB)$. When $\angle AOB$ is greater than 90 degrees (or $\frac{\pi}{4 \text{ radians}}$), and point B extends beyond the boundaries of the positive x-axis and positive y-axis, the values of $\cos(\angle AOB)$ and $\sin(\angle AOB)$ can be expressed as negative values depending on the coordinates of point B. For any angle AOB, place angle AOB in standard position within the circle of radius 1 centered at the origin, with side OA along the positive x-axis and terminal side OB intersecting the circle at point B. Then the cosine of angle AOB is the x-coordinate of B, and the sine of angle AOB is the y-coordinate of B. The tangent of angle AOB is the cosine of angle AOB divided by the sine of angle AOB.

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To convert from degrees to radians, multiply the number of degrees by 2π / 360 degrees. To convert from radians to degrees, multiply the number of radians by 360 degrees / 2π.

An angle with a full rotation about point O has measure 360°. This angle intercepts the full circumference of the circle, which has length 2π. Thus,

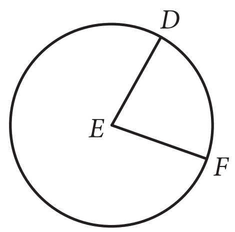
$\frac{\text{measure of an angle in radians}}{\text{measure of an angle in degrees}} = \frac{2\pi}{360°}$. It follows that

measure of an angle in radians $= \frac{2\pi}{360°} \times$ measure of an angle in degrees and

measure of an angle in degrees $= \frac{360°}{2\pi} \times$ measure of an angle in radians.

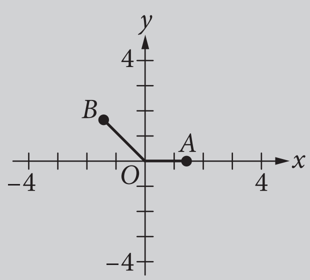
Also note that since a rotation of 2π about point O brings you back to the same point on the unit circle, $\sin(s + 2\pi) = \sin(s)$, $\cos(s + 2\pi) = \cos(s)$, and $\tan(s + 2\pi) = \tan(s)$, for any radian measure s.

Let angle DEF be a central angle in a circle of radius r, as shown in the following figure.



A circle of radius r is similar to a circle of radius 1, with constant of proportionality equal to r. Thus, the length s of the arc intercepted by angle DEF is r times the length of the arc that would be intercepted by an angle of the same measure in a circle of radius 1. Therefore, in the figure above, $s = r \times$ (radian measure of angle DEF), or $\angle DEF = \frac{s}{r}$.

EXAMPLE 10



In the figure above, the coordinates of point B are $(-\sqrt{2}, \sqrt{2})$. What is the measure, in radians, of angle AOB?

A) $\frac{\pi}{4}$

B) $\frac{\pi}{2}$

C) $\frac{3\pi}{4}$

D) $\frac{5\pi}{4}$

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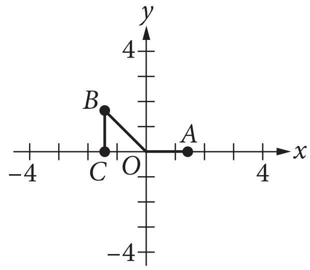
Always be on the lookout for special right triangles. Here, noticing that segment OB is the hypotenuse of a 45-45-90 triangle makes this question easier to solve.

Let C be the point $(-\sqrt{2}, 0)$. Then triangle BOC, shown in the figure below, is a right triangle with both legs of length $\sqrt{2}$.



Hence, triangle BOC is a 45°-45°-90° triangle. Thus, angle COB has measure 45°, and angle AOB has measure $180° - 45° = 135°$. Therefore, the measure of angle AOB in radians is $135° \times \frac{2\pi}{360°} = \frac{3\pi}{4}$, which is choice C.

EXAMPLE 11

$$\sin(x) = \cos(K - x)$$

In the equation above, the angle measures are in radians and K is a constant. Which of the following could be the value of K?

A) 0

B) $\frac{\pi}{4}$

C) $\frac{\pi}{2}$

D) π

REMEMBER

The number i is defined to be the solution to equation $x^2 = -1$. Thus, $i^2 = -1$, and $i = \sqrt{(-1)}$.

The complementary angle relationship for sine and cosine implies that the equation $\sin(x) = \cos(K - x)$ holds if $K = 90°$. Since $90° = \frac{2\pi}{360°} \times 90° = \frac{\pi}{2}$ radians, the value of K could be $\frac{\pi}{2}$, which is choice C.

Complex Numbers

The SAT Math Test may have questions on the arithmetic of complex numbers.

The square of any real number is nonnegative. The number i is defined to be the solution to the equation $x^2 = -1$. That is, $i^2 = -1$, or $i = \sqrt{-1}$. Note that $i^3 = i^2(i) = -i$ and $i^4 = i^2(i^2) = -1(-1) = 1$.

A complex number is a number of the form $a + bi$, where a and b are real numbers and $i = \sqrt{-1}$. This is called the standard form of a complex number. The number a is called the real part of $a + bi$, and the number bi is called the imaginary part of $a + bi$.

Addition and subtraction of complex numbers are performed by adding their real and complex parts. For example,

- $(-3 - 2i) + (4 - i) = (-3 + 4) + (-2i + (-i)) = 1 - 3i$
- $(-3 - 2i) - (4 - i) = (-3 - 4) + (-2i - (-i)) = -7 - i$

Multiplication of complex numbers is performed similarly to multiplication of binomials, using the fact that $i^2 = -1$. For example,

$$\begin{aligned}(-3 - 2i)(4 - i) &= (-3)(4) + (-3)(-i) + (-2i)(4) + (-2i)(-i)\\ &= -12 + 3i - 8i + (-2)(-1)i^2\\ &= -12 - 5i + 2i^2\\ &= -12 - 5i + 2(-1)\\ &= -14 - 5i\end{aligned}$$

The complex number $a - bi$ is called the conjugate of $a + bi$. The product of $a + bi$ and $a - bi$ is $a^2 - bi + bi - b^2i^2$; this reduces to $a^2 + b^2$, a real number. The fact that the product of a complex number and its conjugate is a real number can be used to perform division of complex numbers.

$$\begin{aligned}\frac{-3 - 2i}{4 - i} &= \frac{-3 - 2i}{4 - i} \times \frac{4 + i}{4 + i}\\ &= \frac{(-3 - 2i)(4 + i)}{(4 - i)(4 + i)}\\ &= \frac{-12 - 3i - 8i - 2i^2}{4^2 - i^2}\\ &= \frac{-10 - 11i}{17}\\ &= -\frac{10}{17} - \frac{11}{17}i\end{aligned}$$

EXAMPLE 12

Which of the following is equal to $\frac{1 + i}{1 - i}$?

A) i
B) $2i$
C) $-1 + i$
D) $1 + i$

REMEMBER

If you have little experience working with complex numbers, practice adding, subtracting, multiplying, and dividing complex numbers until you are comfortable doing so. You may see complex numbers on the SAT Math Test.

Multiply both the numerator and denominator of $\frac{1+i}{1-i}$ by $1+i$ to get rid of i from the denominator.

$$\begin{aligned}\frac{1+i}{1-i} &= \frac{1+i}{1-i} \times \frac{1+i}{1+i} \\ &= \frac{(1+i)(1+i)}{(1-i)(1+i)} \\ &= \frac{1+2i+i^2}{1^2-i^2} \\ &= \frac{1+2i-1}{1-(-1)} \\ &= \frac{2i}{2} \\ &= i\end{aligned}$$

Choice A is the correct answer.