Chapter 21

Passport to Advanced Math

Passport to Advanced Math questions include topics that are especially important for students to master *before* studying advanced math. Chief among these topics is the understanding of the structure of expressions and the ability to analyze, manipulate, and rewrite these expressions. This section also includes reasoning with more complex equations and interpreting and building functions. Passport to Advanced Math is one of the three subscores in the SAT Math Test that are reported on a scale of 1 to 15. Questions in this section may be part of the Science subscore or part of the History and Social Studies subscore.

REMEMBER

16 of the 58 questions (28%) on the SAT Math Test are Passport to Advanced Math questions.

As you saw in Chapter 19, the questions in Heart of Algebra focus on the mastery of linear equations, systems of linear equations, and linear functions. In contrast, the questions in Passport to Advanced Math focus on the ability to work with and analyze more complex equations. The questions may require you to demonstrate procedural skill in adding, subtracting, and multiplying polynomials and in dividing a polynomial by a linear expression. You may be required to work with expressions involving exponentials, integer and rational powers, radicals, or fractions with a variable in the denominator. The questions may ask you to solve a quadratic equation, a radical equation, a rational equation, or a system consisting of a linear equation and a nonlinear equation. You may be required to manipulate an equation in several variables to isolate a quantity of interest.

Some questions in Passport to Advanced Math will ask you to build a quadratic or exponential function or an equation that describes a context or to interpret the function or solution to the equation in terms of the context.

Throughout the section, your ability to recognize structure is assessed. Expressions and equations that appear complex may use repeated terms or repeated expressions. By noticing these patterns, the complexity of a problem can be quickly simplified. Structure may be used to factor or otherwise rewrite an expression, to solve a quadratic or other equation, or to draw conclusions about the context represented by an expression, equation, or function. You may be asked to identify or derive the form of an expression or function that reveals information about the expression or function or the context it represents.

Passport to Advanced Math questions also assess your understanding of functions and their graphs. A question may require you to demonstrate your understanding of function notation, including interpreting an expression where the argument of a function is an expression rather than a variable. The questions may assess your knowledge of the domain and range of a function and your understanding of how the algebraic properties of a function relate to the geometric characteristics of its graph.

The questions in this section include both multiple-choice questions and student-produced response questions. On some questions, the use of a calculator is not permitted; on other questions, the use of a calculator is allowed. On questions where the use of a calculator is permitted, you must decide whether using your calculator is an effective strategy.

Let's consider the content and skills assessed by Passport to Advanced Math questions.

Operations with Polynomials and Rewriting Expressions

Questions on the SAT Math Test may assess your ability to add, subtract, and multiply polynomials.

EXAMPLE 1

$$(x^2 + bx - 2)(x + 3) = x^3 + 6x^2 + 7x - 6$$

In the equation above, b is a constant. If the equation is true for all values of x, what is the value of b?

A) 2
B) 3
C) 7
D) 9

REMEMBER

Passport to Advanced Math questions build on the knowledge and skills tested on Heart of Algebra questions. Develop proficiency with Heart of Algebra questions before tackling Passport to Advanced Math questions.

To find the value of b, expand the left-hand side of the equation and then collect like terms so that the left-hand side is in the same form as the right-hand side.

$$\begin{aligned}(x^2 + bx - 2)(x + 3) &= (x^3 + bx^2 - 2x) + (3x^2 + 3bx - 6)\\ &= x^3 + (3 + b)x^2 + (3b - 2)x - 6\end{aligned}$$

Since the two polynomials are equal for all values of x, the coefficient of matching powers of x should be the same. Therefore, $x^3 + (3 + b)x^2 + (3b - 2)x - 6$ and $x^3 + 6x^2 + 7x - 6$ reveals that $3 + b = 6$ and $3b - 2 = 7$. Solving either of these equations gives $b = 3$, which is choice B.

Questions may also ask you to use structure to rewrite expressions. The expression may be of a particular type, such as a difference of squares, or it may require insightful analysis.

EXAMPLE 2

Which of the following is equivalent to $16s^4 - 4t^2$?

A) $4(s^2 - t)(4s^2 + t)$
B) $4(4s^2 - t)(s^2 + t)$
C) $4(2s^2 - t)(2s^2 + t)$
D) $(8s^2 - 2t)(8s^2 + 2t)$

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Passport to Advanced Math questions require a high comfort level working with quadratic equations and expressions, including foiling and factoring. Recognizing classic quadratics such as $x^2 - y^2 = (x - y)(x + y)$ can also improve your speed and accuracy.

This example appears complex at first, but it is very similar to the equation $x^2 - y^2$ and this factors as $(x - y)(x + y)$. The expression $16s^4 - 4t^2$ is also the difference of two squares: $16s^4 - 4t^2 = (4s^2)^2 - (2t)^2$. Therefore, it can be factored as $(4s^2)^2 - (2t)^2 = (4s^2 - 2t)(4s^2 + 2t)$. This expression can be rewritten as $(4s^2 - 2t)(4s^2 + 2t) = 2(2s^2 - t)(2)(2s^2 + t) = 4(2s^2 - t)(2s^2 + t)$, which is choice C.

EXAMPLE 3

$$y^5 - 2y^4 - cxy + 6x$$

In the polynomial above, c is a constant. If the polynomial is divisible by $y - 2$, what is the value of c?

If the expression is divisible by $y - 2$, then the expression $y - 2$ can be factored from the larger expression. Since $y^5 - 2y^4 = (y - 2)y^4$, you have $y^5 - 2y^4 - cxy + 6x = (y - 2)(y^4) - cxy + 6x$. If this entire expression is divisible by $y - 2$, then $-cxy + 6x$ must be divisible by $y - 2$. Thus, $-cxy + 6x = (y - 2)(-cx) = -cxy + 2cx$. Therefore, $2c = 6$, and the value of c is 3.

Quadratic Functions and Equations

Questions in Passport to Advanced Math may require you to build a quadratic function or an equation to represent a context.

EXAMPLE 4

A car is traveling at x feet per second. The driver sees a red light ahead, and after 1.5 seconds reaction time, the driver applies the brake. After the brake is applied, the car takes $\frac{x}{24}$ seconds to stop, during which time the average speed of the car is $\frac{x}{2}$ feet per second. If the car travels 165 feet from the time the driver saw the red light to the time it comes to a complete stop, which of the following equations can be used to find the value of x?

A) $x^2 + 48x - 3{,}960$

B) $x^2 + 48x - 7{,}920$

C) $x^2 + 72x - 3{,}960$

D) $x^2 + 72x - 7{,}920$

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Example 4 requires careful translation of a word problem into an algebraic equation. It pays to be deliberate and methodical when translating word problems into equations on the SAT.

During the 1.5-second reaction time, the car is still traveling at x feet per second, so it travels a total of $1.5x$ feet. The average speed of the car during the $\frac{x}{24}$-second braking interval is $\frac{x}{2}$ feet per second, so over this interval, the car travels $\left(\frac{x}{2}\right)\left(\frac{x}{24}\right) = \frac{x^2}{48}$ feet. Since the total distance the car travels from the time the driver saw the red light to the time it comes to a complete stop is 165 feet, you have the equation $\frac{x^2}{48} + 1.5x = 165$. This quadratic equation can be rewritten in standard form by subtracting 165 from each side and then multiplying each side by 48, giving $x^2 + 72x - 7{,}920$, which is choice D.

REMEMBER

The SAT Math Test may ask you to solve a quadratic equation. Be prepared to use the appropriate method. Practice using the various methods (below) until you are comfortable with all of them.

1. Factoring
2. Completing the square
3. Quadratic formula
4. Using a calculator (if permitted)

Some questions on the SAT Math Test will ask you to solve a quadratic equation. You must determine the appropriate procedure: factoring, completing the square, the quadratic formula, use of a calculator (if permitted), or use of structure. You should also know the following facts in addition to the formulas in the directions:

- The sum of the solutions of $x^2 + bx + c = 0$ is $-b$.
- The product of the solutions of $x^2 + bx + c = 0$ is c.

Each of the facts can be seen from the factored form of a quadratic. If r and s are the solutions of $x^2 + bx + c = 0$, then $x^2 + bx + c = (x - r)(x - s)$. Thus, $b = -(r + s)$ and $c = (-r)(-s)$.

EXAMPLE 5

What are the solutions x of $x^2 - 3 = x$?

A) $\frac{-1 \pm \sqrt{11}}{2}$

B) $\frac{-1 \pm \sqrt{13}}{2}$

C) $\frac{1 \pm \sqrt{11}}{2}$

D) $\frac{1 \pm \sqrt{13}}{2}$

The equation can be solved by using the quadratic formula or by completing the square. Let's use the quadratic formula. First, subtract x from each side of $x^2 - 3 = x$ to put it in standard form: $x^2 - x - 3 = 0$. The quadratic formula states the solutions x of the equation $ax^2 + bx + c = 0$ are $\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$. For the equation $x^2 - x - 3 = 0$, you have $a = 1$, $b = -1$, and $c = -3$. Substituting these formulas into the quadratic formula gives $x = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(1)(-3)}}{2(1)} = \frac{1 \pm \sqrt{1 - (-12)}}{2} = \frac{1 \pm \sqrt{13}}{2}$, which is choice D.

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The quadratic formula states that the solutions x of the equation $ax^2 + bx + c = 0$ are $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

EXAMPLE 6

If $x > 0$ and $2x^2 + 3x - 2 = 0$, what is the value of x?

The left-hand side of the equation can be factored: $2x^2 + 3x - 2 = (2x - 1)(x + 2) = 0$. Therefore, either $2x - 1 = 0$, which gives $x = \frac{1}{2}$, or $x + 2 = 0$, which gives $x = -2$. Since $x > 0$, the value of x is $\frac{1}{2}$.

REMEMBER

Pay close attention to all of the details in the question. In Example 6, x can equal $\frac{1}{2}$ or -2, but since the question states that $x > 0$, the value of x must be $\frac{1}{2}$.

EXAMPLE 7

What is the sum of the solutions of $(2x - 1)^2 = (x + 2)^2$?

If a and b are real numbers and $a^2 = b^2$, then either $a = b$ or $a = -b$. Since $(2x - 1)^2 = (x + 2)^2$, either $2x - 1 = x + 2$ or $2x - 1 = -(x + 2)$. In the first case, $x = 3$, and in the second case, $3x = -1$, or $x = -\frac{1}{3}$. Therefore, the sum of the solutions x of $(2x - 1)^2 = (x + 2)^2$ is $3 + \left(-\frac{1}{3}\right) = \left(\frac{8}{3}\right)$.

Exponential Functions, Equations, and Expressions and Radicals

We examined exponential functions in Examples 7 and 8 of Chapter 20. Some questions in Passport to Advanced Math ask you to build a function that models a given context. As discussed in Chapter 20, exponential functions model situations in which a quantity is multiplied by a constant factor for each time period. An exponential function can be increasing with time, in which case it models exponential growth, or it can be decreasing with time, in which case it models exponential decay.

EXAMPLE 8

A researcher estimates that the population of a city is declining at an annual rate of 0.6%. If the current population of the city is 80,000, which of the following expressions appropriately models the population of the city t years from now according to the researcher's estimate?

A) $80{,}000(1 - 0.006)^t$
B) $80{,}000(1 - 0.006^t)$
C) $80{,}000 - 1.006^t$
D) $80{,}000(0.006^t)$

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A quantity that grows or decays by a fixed percent at regular intervals is said to possess exponential growth or decay.

Exponential growth is represented by the function $y = a(1 + r)^t$, while exponential decay is represented by the function $y = a(1 - r)^t$, where y is the new population, a is the initial population, r is the rate of growth or decay, and t is the number of time intervals that have elapsed.

According to the researcher's estimate, the population is decreasing by 0.6% each year. Since 0.6% is equal to 0.006, after the first year, the population is $80{,}000 - 0.006(80{,}000) = 80{,}000(1 - 0.006)$. After the second year, the population is $80{,}000(1 - 0.006) - 0.006(80{,}000)(1 - 0.006) = 80{,}000(1 - 0.006)^2$. Similarly, after t years, the population will be $80{,}000(1 - 0.006)^t$ according to the researcher's estimate. This is choice A.

Another well-known example of exponential decay is the decay of a radioactive isotope. One example is iodine-131, a radioactive isotope used in some medical treatments. The decay of iodine-131 emits beta and gamma radiation, and it decays to xenon-131. The half-life of iodine-131 is 8.02 days; that is, after 8.02 days, half of the iodine-131 in a sample will have decayed to xenon-131. Suppose a sample of A milligrams of iodine-131 decays for d days. Every 8.02 days, the quantity of iodine-131 is multiplied by $\frac{1}{2}$, or 2^{-1}. In d days, a total of $\frac{d}{8.02}$ different 8.02-day periods will have passed, and so the original quantity will have been multiplied by 2^{-1} a total of $\frac{d}{8.02}$ times. Therefore, the amount, in milligrams, of iodine-131 remaining in the sample will be $A(2^{-1})^{\frac{d}{8.02}} = A\left(2^{-\frac{d}{8.02}}\right)$.

In the preceding discussion, we used the identity $\frac{1}{2} - 2^{-1}$. Questions on the SAT Math Test may require you to apply this and other laws of exponents and the relationship between powers and radicals.

EXAMPLE 9

Which of the following is equivalent to $\left(\frac{1}{\sqrt{x}}\right)^n$?

A) $x^{\frac{n}{2}}$
B) $x^{-\frac{n}{2}}$
C) $x^{n+\frac{1}{2}}$
D) $x^{n-\frac{1}{2}}$

The square root $\sqrt{x}$ is equal to $x^{\frac{1}{2}}$. Thus, $\frac{1}{\sqrt{x}} = x^{-\frac{1}{2}}$, and $\left(\frac{1}{\sqrt{x}}\right)^n = \left(x^{-\frac{1}{2}}\right)^n = x^{-\frac{n}{2}}$. Choice B is the correct answer.

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Practice your exponent rules. Know, for instance, that $\sqrt{x} = x^{\frac{1}{2}}$ and that $\frac{1}{\sqrt{x}} = x^{-\frac{1}{2}}$.

An SAT Math Test question may also ask you to solve a radical equation. In solving radical equations, you may square both sides of an equation. Since squaring is *not* a reversible operation, you may end up with an extraneous root, that is, a root to the simplified equation that is *not* a root to the original equation. Thus, when solving a radical equation, you should check any solution you get in the original equation.

EXAMPLE 10

$$x - 12 = \sqrt{x + 44}$$

What is the solution set for the above equation?

A) {5}
B) {20}
C) {−5, 20}
D) {5, 20}

Squaring each side of $x - 12 = \sqrt{x + 44}$ gives

$$(x - 12)^2 = (\sqrt{x + 44})^2 = x + 44$$
$$x^2 + 24x + 144 = x + 44$$
$$x^2 - 25x + 100 = 0$$
$$(x - 5)(x - 20) = 0$$

The solutions to the quadratic are $x = 5$ and $x = 20$. However, since the first step was to square each side of the given equation, which is not a reversible operation, you need to check $x = 5$ and $x = 20$ in the original equation. Substituting 5 for x gives

$$5 - 12 = \sqrt{5 + 44}$$
$$-7 = \sqrt{49}$$

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A good strategy to use when solving radical equations is to square both sides of the equation. When doing so, however, be sure to check the solutions in the original equation, as you may end up with a root that is not a solution to the original equation.

This is not a true statement (since $\sqrt{49}$ represents only the positive square root, 7), so $x = 5$ is *not* a solution to $x - 12 = \sqrt{x + 44}$. Substituting 20 for x gives

$$20 - 12 = \sqrt{20 + 44}$$
$$8 = \sqrt{64}$$

This is a true statement, so $x = 20$ is a solution to $x - 12 = \sqrt{x + 44}$. Therefore, the solution set is {20}, which is choice B.

Dividing Polynomials by a Linear Expression and Solving Rational Equations

Questions on the SAT Math Test may assess your ability to work with rational expressions, including fractions with a variable in the denominator. This may include long division of a polynomial by a linear expression or finding the solution to a rational equation.

EXAMPLE 11

When $6x^2 - 5x + 4$ is divided by $3x + 2$, the result is $2x - 3 + \frac{R}{(3x + 2)}$, where R is a constant. What is the value of R?

Performing the long division gives

$$\begin{array}{r|l} & 2x - 3 \\ 3x + 2 & 6x^2 - 5x + 4 \\ & \underline{6x^2 + 4x} \\ & \quad -9x + 4 \\ & \quad \underline{-9x - 6} \\ & \qquad 10 \end{array}$$

Therefore, the remainder is 10.

If $ax + b$ is a factor of the polynomial $P(x)$, then $P(x)$ can be written as

$$P(x) = (ax + b)Q(x)$$

for some polynomial $Q(x)$. It follows that the solution to $ax + b = 0$, namely, $x = -\frac{b}{a}$, is a solution to $P(x) = 0$. More generally, if the number r is the remainder when $P(x)$ is divided by $ax + b$, you have

$$P(x) = (ax + b)Q(x) + r$$

It follows that for $x = -\frac{b}{a}$, the value of $P\left(-\frac{b}{a}\right) = (0)(Q(x)) + r = r$. This is another way to solve Example 11. The solution of $3x + 2 = 0$ is $x = -\frac{2}{3}$, so the remainder when $6x^2 - 5x + 4$ is divided by $3x + 2$ is the value of $6x^2 - 5x + 4$ when $-\frac{2}{3}$ is substituted for x: Remainder: $6\left(-\frac{2}{3}\right)^2 - 5\left(-\frac{2}{3}\right) + 4 = \frac{8}{3} + \frac{10}{3} + 4 = 10$.

EXAMPLE 12

$$\frac{3}{t+1} = \frac{2}{t+3} + \frac{1}{4}$$

If t is a solution to the equation above and $t > 0$, what is the value of t?

The first step in solving this equation is to clear the variable out of the denominators by multiplying each side by $(t + 1)(t + 3)$. This gives $3(t + 3) = 2(t + 1) + \frac{1}{4}(t + 1)(t + 3)$. Now multiply each side by 4 to get rid of the fraction: $12(t + 3) = 8(t + 1)(t + 3)$. Expanding all the products and moving all the terms to the right-hand side gives $0 = t^2 - 25$. Therefore, the solutions to the equation are $t = 5$ and $t = -5$. Since $t > 0$, the value of t is 5.

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When solving for a variable in an equation involving fractions, a good first step is to clear the variable out of the denominators of the fractions.

Systems of Equations

Questions on the SAT Math Test may ask you to solve a system of equations in two variables in which one equation is linear and the other equation is quadratic or another nonlinear equation.

EXAMPLE 13

$$3x + y = -3$$
$$(x + 1)^2 - 4(x + 1) - 6 = y$$

If (x, y) is a solution of the system of equations above and $y > 0$, what is the value of y?

The structure of the second equation suggests that $(x + 1)$ is a factor of the first equation. Subtracting $3x$ from each side of the first equation gives you $y = -3 - 3x$, which can be rewritten as $y = -3(x + 1)$. Substituting $-3(x + 1)$ for y in the second equation gives you $(x + 1)^2 - 4(x + 1) - 6 = -3(x + 1)$, which can be rewritten as $(x + 1)^2 - (x + 1) - 6 = 0$. The structure of this equation suggests that $x + 1$ can be treated as a variable. Factoring gives you $((x + 1) - 3)((x + 1) + 2) = 0$, or $(x - 2)(x + 3) = 0$. Thus, either $x = 2$, which gives $y = -3 - 3(2) = -9$; or $x = -3$, which gives $y = -3 - 3(-3) = -9$. Therefore, the solutions to the system are $(2, -9)$ and $(-3, 6)$. Since the question states that $y > 0$, the value of y is 6.

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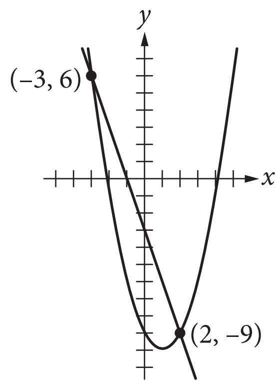
The first step to solving this example is substitution, an approach you may use on Heart of Algebra questions. The other key was noticing that $(x + 1)$ can be treated as a variable.



The solutions of the system are given by the intersection points of the two graphs. Questions on the SAT Math Test may assess this or other relationships between algebraic and graphical representations of functions.

Relationships Between Algebraic and Graphical Representations of Functions

A function $f(x)$ has a graph in the xy-plane, which is the graph of the equation $y = f(x)$ (or, equivalently, consists of all ordered pairs $(x, f(x))$). Some questions in Passport to Advanced Math assess your ability to relate properties of the function f to properties of its graph, and vice versa. You may be required to apply some of the following relationships:

- **Intercepts.** The x-intercepts of the graph of f correspond to values of x such that $f(x) = 0$; if the function f has no zeros, its graph has no x-intercepts, and vice versa. The y-intercept of the graph of f corresponds to the value of $f(0)$. If $x = 0$ is not in the domain of f, the graph of f has no y-intercept, and vice versa.

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The domain of a function is the set of all values for which the function is defined. The range of a function is the set of all values that are the output, or result, of applying the function.

- **Domain and range.** The domain of f is the set of all x for which $f(x)$ is defined. The range of f is the set of all y with $y = f(x)$ for some value of x in the domain. The domain and range can be found from the graph of f as the set of all x-coordinates and y-coordinates, respectively, of points on the graph.

- **Maximum and minimum values.** The maximum and minimum values of f can be found by locating the highest and the lowest points on the graph, respectively. For example, suppose P is the highest point on the graph of f. Then the y-coordinate of P is the maximum value of f, and the x-coordinate of P is where f takes on its maximum value.

- **Increasing and decreasing.** The graph of f shows the intervals over which the function f is increasing and decreasing.

- **End behavior.** The graph of f can indicate if $f(x)$ increases or decreases without limit as x gets very large and positive or very large and negative.

- **Asymptotes.** If the values of f approach a fixed value, say K, as x gets very large and positive or very large and negative, the graph of f has a horizontal asymptote at $y = K$. If f is a rational function whose denominator is zero and numerator is nonzero at $x = a$, then the graph of f has a vertical asymptote at $x = a$.

- **Symmetry.** If the graph of f is symmetric about the y-axis, then f is an even function, that is, $f(-x) = f(x)$ for all x in the domain of f. If the graph of f is symmetric about the origin, then f is an odd function, that is, $f(-x) = -f(x)$ for all x in the domain of f.

- **Transformations.** For a graph of a function f, a change of the form $f(x) + a$ will result in a vertical shift of a units and a change of the form $f(x + a)$ will result in a horizontal shift of a units.

Note: The SAT Math Test uses the following conventions about graphs in the xy-plane *unless* a particular question clearly states or shows a different convention:

- The axes are perpendicular.
- Scales on the axes are linear scales.
- The size of the units on the two axes *cannot* be assumed to be equal unless the question states they are equal or you are given enough information to conclude they are equal.
- The values on the horizontal axis increase as you move to the right.
- The values on the vertical axis increase as you move up.

REMEMBER

Don't assume the size of the units on the two axes are equal unless the question states they are equal or you can conclude they are equal from the information given.

EXAMPLE 14

The graph of which of the following functions in the xy-plane has x-intercepts at -4 and 5?

A) $f(x) = (x + 4)(x - 5)$
B) $g(x) = (x - 4)(x + 5)$
C) $h(x) = (x - 4)^2 + 5$
D) $k(x) = (x + 5)^2 - 4$

The x-intercepts of the graph of a function correspond to the zeros of the function. If a function has x-intercepts at -4 and 5, then the values of the function at -4 and 5 are each 0. The function in choice A is in factored form,

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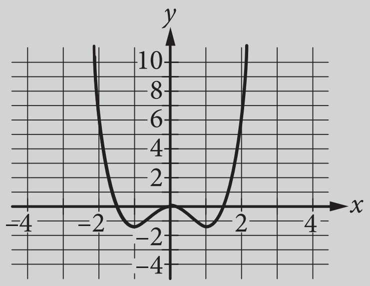
Another way to think of this question is to ask yourself, "Which answer choice represents a function that has values of zero when $x = -4$ and $x = +5$?"

which shows that $f(x) = 0$ if and only if $x + 4 = 0$ or $x - 5 = 0$, that is, if $x = -4$ or $x = 5$. Therefore, $f(x) = (x + 4)(x - 5)$ has x-intercepts at −4 and 5.

The graph in the xy-plane of each of the functions in the previous example is a parabola. Using the defining equations, you can tell that the graph of g has x-intercepts at 4 and −5; the graph of h has its vertex at (4, 5); and the graph of k has its vertex at (−5, −4).

EXAMPLE 15



The function $f(x) = x^4 - 2.4x^2$ is graphed in the xy-plane as shown above. If k is a constant such that the equation $f(x) = k$ has 4 solutions, which of the following could be the value of k?

A) 1
B) 0
C) −1
D) −2

Choice C is correct. The equation $f(x) = k$ will have 4 solutions if and only if the graph of the horizontal line with equation $y = k$ intersects the graph of f at 4 points. The graph shows that of the given choices, only for choice C, −1, does the graph of $y = -1$ intersect the graph of f at 4 points.

Function Notation

The SAT Math Test assesses your understanding of function notation. You must be able to evaluate a function given the rule that defines it, and if the function describes a context, you may need to interpret the value of the function in the context. A question may ask you to interpret a function when an expression, such as $2x$ or $x + 1$, is used as the argument instead of the variable x, or a question may ask you to evaluate the composition of two functions.

EXAMPLE 16

If $g(x) = 2x + 1$ and $f(x) = g(x) + 4$, what is $f(2)$?

You are given $f(x) = g(x) + 4$ and therefore $f(2) = g(2) + 4$. To determine the value of $g(2)$, use the function $g(x) = 2x + 1$. Thus, $g(2) = 2(2) + 1$, and $g(2) = 5$. Substituting $g(2)$ gives $f(2) = 5 + 4$, or $f(2) = 9$.

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What may seem at first to be a complex question boils down to straightforward substitution.

Analyzing More Complex Equations in Context

Equations and functions that describe a real-life context can be complex. Often it is not possible to analyze them as completely as you can analyze a linear equation or function. You still can acquire key information about the context by analyzing the equation or function that describes it. Questions on the Passport to Advanced Math section may ask you to use an equation describing a context to determine how a change in one quantity affects another quantity. You may also be asked to manipulate an equation to isolate a quantity of interest on one side of the equation. You may be asked to produce or identify a form of an equation that reveals new information about the context it represents or about the graphical representation of the equation.

EXAMPLE 17

If an object of mass m is moving at speed v, the object's kinetic energy KE is given by the equation $\text{KE} = \frac{1}{2}mv^2$. If the mass of the object is halved and its speed is doubled, how does the kinetic energy change?

A) The kinetic energy is halved.
B) The kinetic energy is unchanged.
C) The kinetic energy is doubled.
D) The kinetic energy is quadrupled (multiplied by a factor of 4).

Choice C is correct. If the mass of the object is halved, the new mass is $\frac{m}{2}$. If the speed of the object is doubled, its new speed is $2v$. Therefore, the new kinetic energy is $\frac{1}{2}\left(\frac{m}{2}\right)(2v)^2 = \frac{1}{2}\left(\frac{m}{2}\right)(4v^2) = mv^2$. This is double the kinetic energy of the original object, which was $\frac{1}{2}mv^2$.

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Another way to check your answer here is to pick simple numbers for mass and speed and examine the impact on kinetic energy when those values are altered as indicated by the question. If mass and speed both equal 1, kinetic energy is $\frac{1}{2}$. When mass is halved, to $\frac{1}{2}$, and speed is doubled, to 2, the new kinetic energy is 1. Since 1 is twice the value of $\frac{1}{2}$, we know that kinetic energy is doubled.

EXAMPLE 18

A gas in a container will escape through holes of microscopic size, as long as the holes are larger than the gas molecules. This process is called effusion. If a gas of molar mass M_1 effuses at a rate of r_1 and a gas of molar mass M_2 effuses at a rate of r_2, then the following relationship holds.

$$\frac{r_1}{r_2} = \sqrt{\frac{M_2}{M_1}}$$

This is known as Graham's law. Which of the following correctly expresses M_2 in terms of M_1, r_1, and r_2?

A) $M_2 = M_1\frac{r_1^2}{r_2^2}$

B) $M_2 = M_1\frac{r_2^2}{r_1^2}$

C) $M_2 = \sqrt{M_1}\frac{r_1}{r_2}$

D) $M_2 = \sqrt{M_1}\frac{r_2}{r_1}$

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Always start by identifying exactly what the question asks. In this case, you are being asked to isolate the variable M_2. Squaring both sides of the equation is a great first step as it allows you to eliminate the radical sign.

Squaring each side of $\frac{r_1}{r_2} = \sqrt{\frac{M_2}{M_1}}$ gives $\left(\frac{r_1}{r_2}\right)^2 = \left(\sqrt{\frac{M_2}{M_1}}\right)^2$, which can be rewritten as $\frac{M_2}{M_1} = \frac{r_1^2}{r_2^2}$. Multiplying each side of $\frac{M_2}{M_1} = \frac{r_1^2}{r_2^2}$ by M_1 gives $M_2 = M_1\frac{r_1^2}{r_2^2}$, which is choice A.

EXAMPLE 19

A store manager estimates that if a video game is sold at a price of p dollars, the store will have weekly revenue, in dollars, of $r(p) = -4p^2 + 200p$ from the sale of the video game. Which of the following equivalent forms of $r(p)$ shows, as constants or coefficients, the maximum possible weekly revenue and the price that results in the maximum revenue?

A) $r(p) = 200p - 4p^2$
B) $r(p) = -2(2p^2 - 100p)$
C) $r(p) = -4(p^2 - 50p)$
D) $r(p) = -4(p - 25)^2 + 2{,}500$

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The fact that the coefficient of the squared term is negative for this function indicates that the graph of r in the coordinate plane is a parabola that opens downward. Thus, the maximum value of revenue corresponds to the vertex of the parabola.

Choice D is correct. The graph of r in the coordinate plane is a parabola that opens downward. The maximum value of revenue corresponds to the vertex of the parabola. Since the square of any real number is always nonnegative, the form $r(p) = -4(p - 25)^2 + 2{,}500$ shows that the vertex of the parabola is (25, 2,500); that is, the maximum must occur where $-4(p - 25)^2$ is 0, which is $p = 25$, and this maximum is $r(25) = 2{,}500$. Thus, the maximum possible weekly revenue and the price that results in the maximum revenue occur as constants in the form $r(p) = -4(p - 25)^2 + 2{,}500$.