Chapter 19

Heart of Algebra

Heart of Algebra focuses on the mastery of linear equations, systems of linear equations, and linear functions. The ability to analyze and create linear equations, inequalities, and functions is essential for success in college and careers, as is the ability to solve linear equations and systems fluently. The questions in Heart of Algebra include both multiple-choice questions and student-produced response questions. On some questions, the use of a calculator is not permitted; on other questions, the use of a calculator is allowed.

The questions in Heart of Algebra vary significantly in form and appearance. They may be straightforward fluency exercises or pose challenges of strategy or understanding, such as interpreting the relationship between graphical and algebraic representations or solving as a process of reasoning. You will be required to demonstrate both procedural skill and a deep understanding of concepts.

Let's explore the content and skills assessed by Heart of Algebra questions.

REMEMBER

The SAT Math Test will require you to demonstrate a deep understanding of several core algebra topics, namely linear equations, systems of linear equations, and linear functions. These topics are fundamental to the learning and work often required in college and career.

Linear Equations, Linear Inequalities, and Linear Functions in Context

When you use algebra to analyze and solve a problem in real life, a key step is to represent the context of the problem algebraically. To do this, you may need to define one or more variables that represent quantities in the context. Then you need to write one or more expressions, equations, inequalities, or functions that represent the relationships described in the context. For example, once you write an equation that represents the context, you solve the equation. Then you interpret the solution to the equation in terms of the context. Questions on the SAT Math Test may assess your ability to accomplish any or all of these steps.

EXAMPLE 1

> In 2014, County X had 783 miles of paved roads. Starting in 2015, the county has been building 8 miles of new paved roads each year. At this rate, how many miles of paved road will County X have in 2030? (Assume that no paved roads go out of service.)

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Many Heart of Algebra questions such as this one will require you to accomplish the following steps:

1. Define one or more variables that represent quantities in the question.
2. Write one or more equations, expressions, inequalities, or functions that represent the relationships described in the question.
3. Solve the equation, and interpret the solution in terms of what the question is asking.

Ample practice with each of these steps will help you develop your math skills and knowledge.

The first step in answering this question is to decide what variable or variables you need to define. The question is asking how the number of miles of paved road in County X depends on the year. This can be represented using n, the number of years after 2014. Then, since the question says that County X had 783 miles of paved road in 2014 and is building 8 miles of new paved roads each year, the expression $783 + 8n$ gives the number of miles of paved roads in County X in year n. The year 2030 is $2030 - 2014 = 16$ years after 2014; thus, the year 2030 corresponds to $n = 16$. Hence, to find the number of miles of paved roads in County X in 2030, substitute 16 for n in the expression $783 + 8n$, giving $783 + 8(16) = 783 + 128 = 911$. Therefore, at the given rate of building, County X will have 911 miles of paved roads in 2030.

There are different questions that can be asked about the same context.

EXAMPLE 2

> In 2014, County X had 783 miles of paved roads. Starting in 2015, the county has been building 8 miles of new paved roads each year. At this rate, if n is the number of years after 2014, which of the following functions f gives the number of miles of paved road there will be in County X? (Assume that no paved roads go out of service.)
>
> A) $f(n) = 8 + 783n$
> B) $f(n) = 2{,}014 + 783n$
> C) $f(n) = 783 + 8n$
> D) $f(n) = 2{,}014 + 8n$

REMEMBER

There are several different ways you can be tested on the same underlying algebra concepts. Practicing a variety of questions, with different contexts, is a good way to ensure you'll be ready for the questions you'll come across on the SAT.

This question already defines the variable and asks you to create a function that describes the context. The discussion in Example 1 shows that the correct answer is choice C.

EXAMPLE 3

> In 2014, County X had 783 miles of paved roads. Starting in 2015, the county has been building 8 miles of new paved roads each year. At this rate, in which year will County X first have at least 1,000 miles of paved roads? (Assume that no paved roads go out of service.)

In this question, you must solve an inequality. As in Example 1, let n be the number of years after 2014. Then the expression $783 + 8n$ gives the number of miles of paved roads in County X. The question is asking when there will first be at least 1,000 miles of paved roads in County X. This condition can be represented by the inequality $783 + 8n \geq 1{,}000$. To find the year in which there will first be at least 1,000 miles of paved roads, you solve this inequality for n. Subtracting 783 from each side of $783 + 8n \geq 1{,}000$ gives $8n \geq 217$. Then dividing each side of $8n \geq 217$ gives $n \geq 27.125$. Note that an important part of relating the inequality $783 + 8n \geq 1{,}000$ back to the context is to notice that n is counting calendar years and so it must be an integer. The least value of n that satisfies $783 + 8n \geq 1{,}000$ is 27.125, but the year $2014 + 27.125 = 2041.125$ does not make sense as an answer, and in 2041, there would be only $783 + 8(27) = 999$ miles of paved roads in the county. Therefore, the variable n needs to be rounded up to the next integer, and so the least possible value of n is 28. Therefore, the year that County X will first have at least 1,000 miles of paved roads is 28 years after 2014, or 2042.

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Solving an equation or inequality is often only part of the problem-solving process. You must also interpret the solution in the context of the question, so be sure to remind yourself of the question's context and the meaning of the variables you solved for before selecting your answer.

In Example 1, once the variable n was defined, you needed to find an expression that represents the number of miles of paved road in terms of n. In other questions, creating the correct expression, equation, or function may require a more insightful understanding of the context.

EXAMPLE 4

To edit a manuscript, Miguel charges \$50 for the first 2 hours and \$20 per hour after the first 2 hours. Which of the following expresses the amount in dollars, C, Miguel charges if it takes him x hours to edit a manuscript, where $x > 2$?

A) $C = 20x$
B) $C = 20x + 10$
C) $C = 20x + 50$
D) $C = 20x + 90$

The question defines the variables C and x and asks you to express C in terms of x. To create the correct expression, you must note that since the \$50 that Miguel charges pays for his first 2 hours of editing, he charges \$20 per hour only *after* the first 2 hours. Thus, if it takes x hours for Miguel to edit a manuscript, he charges \$50 for the first 2 hours and \$20 per hour for the remaining time, which is $x - 2$ hours. Thus, his total charge, C, can be written as $C = 50 + 20(x - 2)$. This does not match any of the choices. But when the right-hand side of $C = 50 + 20(x - 2)$ is expanded, you get $C = 50 + 20x - 40$, or $C = 20x + 10$, which is choice B.

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When the solution you arrive at doesn't match any of the answer choices, consider if expanding, simplifying, or rearranging your solution will cause it to match an answer choice. Often, this extra step is needed to arrive at the correct answer.

As with Examples 1 to 3, there are different questions that could be asked about this context. For example, you could be asked to find how long it took Miguel to edit a manuscript if he charged \$370.

Absolute Value

Absolute value expressions, inequalities, and equations are included in Heart of Algebra. (Graphs of absolute value equations and functions are in Passport to Advanced Math.) One definition of absolute value is

$$|x| = \begin{cases} x, \text{ if } x \geq 0 \\ -x, \text{ if } x < 0 \end{cases}$$

The absolute value of any real number is nonnegative. An important consequence of this definition is that $|-x| = |x|$ for any real number x. Another important consequence of this definition is that if a and b are any two real numbers, then $|a - b|$ is equal to the distance between a and b on the number line.

EXAMPLE 5

REMEMBER

The SAT Math Test will require you to have a deep understanding of absolute value, not simply that the absolute value of any real number is nonnegative. Example 5, for instance, requires you to apply your knowledge of two concepts, absolute value and inequality, to represent the relationship described in a specific context.

The stratosphere is the layer of the Earth's atmosphere that is more than 10 kilometers (km) and less than 50 km above the Earth's surface. Which of the following inequalities describes all possible heights x, in km, above the Earth's surface that are in the stratosphere?

A) $|x + 10| < 50$
B) $|x - 10| < 50$
C) $|x + 30| < 20$
D) $|x - 30| < 20$

The question states that the stratosphere is the layer of the Earth's atmosphere that is greater than 10 km and less than 50 km above the Earth's surface. Thus, the possible heights x, in km, above the Earth's surface that are in the stratosphere are given by the inequality $10 < x < 50$. To answer the question, you need to find an absolute value inequality that is equivalent to $10 < x < 50$.

The inequality $10 < x < 50$ describes the open interval (10, 50). To describe an interval with an absolute value inequality, use the midpoint and the size of the interval. The midpoint of (10, 50) is $\frac{10 + 50}{2} = 30$. Then observe that the interval (10, 50) consists of all points that are within 20 of the midpoint. That is, (10, 50) consists of x, whose distance from 30 on the number line is less than 20. The distance between x and 30 on the number line is $|x - 30|$. Therefore, the possible values of x are described by $|x - 30| < 20$, which is choice D.

Systems of Linear Equations and Inequalities in Context

You may need to define more than one variable and create more than one equation or inequality to represent a context and answer a question. There are questions on the SAT Math Test that require you to create and solve a system of equations or create a system of inequalities.

EXAMPLE 6

> Maizah bought a pair of pants and a briefcase at a department store. The sum of the prices before sales tax was \$130.00. There was no sales tax on the pants and a 9% sales tax on the briefcase. The total Maizah paid, including the sales tax, was \$136.75. What was the price, in dollars, of the pants?

To answer the question, you first need to define the variables. The question discusses the prices of a pair of pants and a briefcase and asks you to find the price of the pants. So it is appropriate to let P be the price of the pants, in dollars, and to let B be the price of the briefcase, in dollars. Since the sum of the prices before sales tax was \$130.00, the equation $P + B = 130$ is true. A sales tax of 9% was added to the price of the briefcase. Since 9% is equal to 0.09, the price of the briefcase with tax was $B + 0.09B = 1.09B$. There was no sales tax on the pants, and the total Maizah paid, including tax, was \$136.75, so the equation $P + 1.09B = 136.75$ holds.

Now, you need to solve the system

$$P + B = 130$$
$$P + 1.09B = 136.75$$

Subtracting the sides of the first equation from the corresponding sides of the second equation gives you $(P + 1.09B) - (P + B) = 136.75 - 130$, which simplifies to $0.09B = 6.75$. Now you can divide each side of $0.09B = 6.75$ by 0.09. This gives you $B = \frac{6.75}{0.09} = 75$. This is the value of B, the price, in dollars, of the briefcase. The question asks for the price, in dollars, of the pants, which is P. You can substitute 75 for B in the equation $P + B = 130$, which gives you $P + 75 = 130$, or $P = 130 - 75 = 55$, so the pants cost \$55.

(Note that this example has no choices. It is a student-produced response question. On the SAT, you would grid your answer in the spaces provided on the answer sheet.)

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You can use either of two approaches — combination or substitution — when solving a system of linear equations. One may get you to the answer more quickly than the other, depending on the equations you're working with and what you're solving for. Practice using both to give you greater flexibility on test day.

REMEMBER

While this question may seem complex, as it involves numerous steps, solving it requires a strong understanding of the same underlying principles outlined above: defining variables, creating equations to represent relationships, solving equations, and interpreting the solution.

EXAMPLE 7

Each morning, John jogs at 6 miles per hour and rides a bike at 12 miles per hour. His goal is to jog and ride his bike a total of at least 9 miles in less than 1 hour. If John jogs j miles and rides his bike b miles, which of the following systems of inequalities represents John's goal?

A) $\frac{j}{6} + \frac{b}{12} < 1$
$j + b \geq 9$

B) $\frac{j}{6} + \frac{b}{12} \geq 1$
$j + b < 9$

C) $6j + 12b \geq 9$
$j + b < 1$

D) $6j + 12b < 1$
$j + b \geq 9$

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In Example 7, the answer choices each contain two parts. Use this to your advantage by tackling one part at a time and eliminating answers that don't work.

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You should be able to quickly rearrange three-part equations such as the rate equation (rate = distance / time) for any of the three parts. Example 7 requires you to solve the equation for time.

John jogs j miles and rides his bike b miles; his goal to jog and ride his bike a total of at least 9 miles is represented by the inequality $j + b \geq 9$. This eliminates choices B and C.

Since rate × time = distance, it follows that time is equal to distance divided by rate. John jogs j miles at 6 miles per hour, so the time he jogs is equal to $\frac{j \text{ miles}}{6 \text{ miles/hour}} = \frac{j}{6}$ hours. Similarly, since John rides his bike b miles at 12 miles per hour, the time he rides his bike is $\frac{b}{12}$ hours. Thus, John's goal to complete his jog and his bike ride in less than 1 hour can be represented by the inequality $\frac{j}{6} + \frac{b}{12} = < 1$. The system $j + b \geq 9$ and $\frac{j}{6} + \frac{b}{12} < 1$ is choice A.

Fluency in Solving Linear Equations, Linear Inequalities, and Systems of Linear Equations

Creating linear equations, linear inequalities, and systems of linear equations that represent a context is a key skill for success in college and careers. It is also essential to be able to fluently solve linear equations, linear inequalities, and systems of linear equations. Some of the questions in the Heart of Algebra section of the SAT Math Test present equations, inequalities, or systems without a context and directly assess your fluency in solving them.

Some fluency questions allow the use of a calculator; other questions do not permit the use of a calculator and test your ability to solve equations, inequalities, and systems of equations by hand. Even for questions where a calculator is allowed, you may be able to answer the question more quickly without using a calculator, such as in Example 9. Part of what the SAT Math Test assesses is your ability to decide when using a calculator to answer a question is appropriate. Example 8 is an example of a question that could appear on either the calculator or no-calculator portion of the Math Test.

EXAMPLE 8

$$3\left(\frac{1}{2} - y\right) = \frac{3}{5} + 15y$$

What is the solution to the equation above?

Expanding the left-hand side of the equation gives $\frac{3}{2} - 3y = \frac{3}{5} + 15y$, which can be rewritten as $18y = \frac{3}{2} - \frac{3}{5}$. Multiplying each side of $18y = \frac{3}{2} - \frac{3}{5}$ by 10, the least common multiple of 2 and 5, clears the denominators: $180y = \frac{30}{2} - \frac{30}{5} = 15 - 6 = 9$. Therefore, $y = \frac{9}{180} = \frac{1}{20}$.

EXAMPLE 9

$$-2(3x - 2.4) = -3(3x - 2.4)$$

What is the solution to the equation above?

You could solve this in the same way as Example 8, by multiplying everything out and simplifying. But the structure of the equation reveals that -2 times a quantity, $3x - 2.4$, is equal to -3 times the same quantity. This is only possible if the quantity $3x - 2.4$ is equal to zero. Thus, $3x - 2.4 = 0$, or $3x = 2.4$. Therefore, the solution is $x = 0.8$.

EXAMPLE 10

$$-2x = 4y + 6$$

$$2(2y + 3) = 3x - 5$$

What is the solution (x, y) to the system of equations above?

This is an example of a system you can solve quickly by substitution. Since $-2x = 4y + 6$, it follows that $-x = 2y + 3$. Now you can substitute $-x$ for $2y + 3$ in the second equation. This gives you $2(-x) = 3x - 5$, which simplifies to $5x = 5$, or $x = 1$. Substituting 1 for x in the first equation gives you $-2 = 4y + 6$, which simplifies to $4y = -8$, or $y = -2$. Therefore, the solution to the system is $(1, -2)$.

REMEMBER

While a calculator is permitted on one portion of the SAT Math Test, it's important to not over-rely on a calculator. Some questions, such as Example 9, can be solved more efficiently without using a calculator. Your ability to choose when to use and when not to use a calculator is one of the things the SAT Math Test assesses, so be sure to practice this in your studies.

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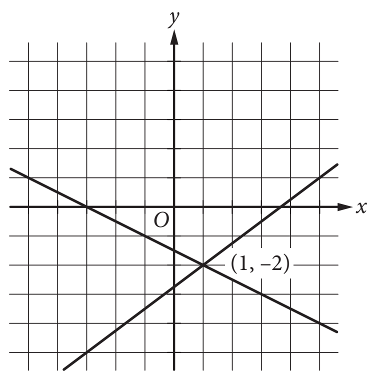
In Example 6, the combination approach yields an efficient solution to the question. In Example 10, substitution turns out to be a fast approach. These examples illustrate the benefits of knowing both approaches and thinking critically about which approach may be faster on a given question.

In the preceding examples, you have found a unique solution to linear equations and to systems of two linear equations in two variables. But not all such equations and systems have solutions, and some have infinitely many solutions. Some questions on the SAT Math Test assess your ability to determine whether an equation or a system has one solution, no solutions, or infinitely many solutions.

The Relationships among Linear Equations, Lines in the Coordinate Plane, and the Contexts They Describe

A system of two linear equations in two variables can be solved by graphing the lines in the coordinate plane. For example, you can graph the system of equations in Example 10 in the *xy*-plane:



The point of intersection gives the solution to the system.

If the equations in a system of two linear equations in two variables are graphed, each graph will be a line. There are three possibilities:

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Graphing systems of two linear equations is another effective approach to solving them. Practice arranging linear equations into $y = mx + b$ form and graphing them in the coordinate plane.

1. The lines intersect in one point. In this case, the system has a unique solution.
2. The lines are parallel. In this case, the system has no solution.
3. The lines are identical. In this case, every point on the line is a solution, and so the system has infinitely many solutions.

By putting the equations in the system into slope-intercept form, the second and third cases can be identified. If the lines have the same slope and different *y*-intercepts, they are parallel; if both the slope and the *y*-intercept are the same, the lines are identical.

How are the second and third cases represented algebraically? Examples 11 and 12 concern this question.

EXAMPLE 11

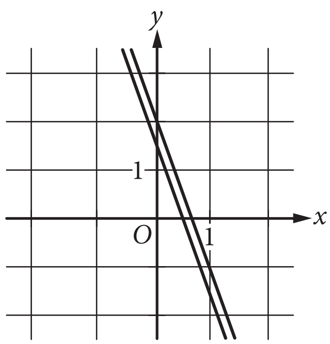
$$2y + 6x = 3$$
$$y + 3x = 2$$

How many solutions (x, y) are there to the system of equations above?

A) Zero
B) One
C) Two
D) More than two

If you multiply each side of $y + 3x = 2$ by 2, you get $2y + 6x = 4$. Then subtracting each side of $2y + 6x = 3$ from the corresponding side of $2y + 6x = 4$ gives $0 = 1$. This is a false statement. Therefore, the system has zero solutions (x, y).

Alternatively, you could graph the two equations. The graphs are parallel lines, so there are no points of intersection.



REMEMBER

When the graphs of a system of two linear equations are parallel lines, as in Example 11, the system has zero solutions. If the question states that a system of two linear equations has an infinite number of solutions, as in Example 12, the equations must be equivalent.

EXAMPLE 12

$$3s - 2t = a$$
$$-15s + bt = -7$$

In the system of equations above, a and b are constants. If the system has infinitely many solutions, what is the value of a?

If a system of two linear equations in two variables has infinitely many solutions, the two equations in the system must be equivalent. Since the two equations are presented in the same form, the second equation must be equal to the first equation multiplied by a constant. Since the coefficient of s in the second equation is -5 times the coefficient of s in the first equation, multiply each side of the first equation by -5. This gives you the system

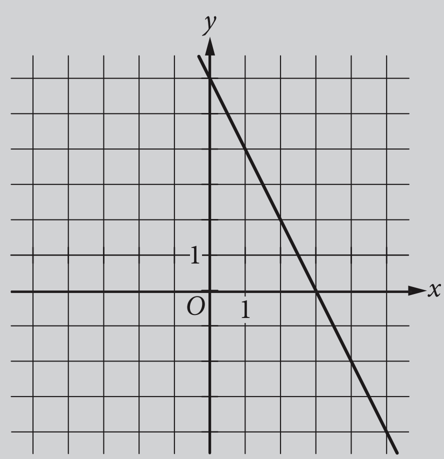
$$-15s + 10t = -5a$$
$$-15s + bt = -7$$

Since these two equations are equivalent and have the same coefficient of s, the coefficients of t and the constants on the right-hand side must also be the same. Thus, $b = 10$ and $-5a = -7$. Therefore, the value of a is $\frac{7}{5}$.

There will also be questions on the SAT Math Test that assess your knowledge of the relationship between the algebraic and the geometric representations of a line, that is, between an equation of a line and its graph. The key concepts are:

- If the slopes of line ℓ and line k are each defined (that is, if neither line is a vertical line), then
    - Line ℓ and line k are parallel if and only if they have the same slope.
    - Line ℓ and line k are perpendicular if and only if the product of their slopes is -1.

EXAMPLE 13



The graph of line k is shown in the xy-plane above. Which of the following is an equation of a line that is perpendicular to line k?

A) $y = -2x + 1$

B) $y = -\frac{1}{2}x + 2$

C) $y = \frac{1}{2}x + 3$

D) $y = 2x + 4$

REMEMBER

The SAT Math Test will further assess your understanding of linear equations by, for instance, asking you to select a linear equation that describes a given graph, select a graph that describes a given linear equation, or determine how a graph may be impacted by a change in its equation.

Note that the graph of line k passes through the points (0, 6) and (3, 0). Thus, the slope of line k is $\frac{0-6}{3-0} = -2$. Since the product of the slopes of perpendicular lines is −1, a line that is perpendicular to line k will have slope $\frac{1}{2}$. All the choices are in slope-intercept form, and so the coefficient of x is the slope of the line represented by the equation. Therefore, choice C, $y = \frac{1}{2}x + 3$, is an equation of a line with slope $\frac{1}{2}$, and thus this line is perpendicular to line k.

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Example 13 requires a strong understanding of slope as well as the ability to calculate slope: slope = rise / run = change in x / change in y. Parallel lines have slopes that are equal. Perpendicular lines have slopes whose product is −1.

As we've noted, some contexts can be described with a linear equation. The graph of a linear equation is a line. A line has geometric properties such as its slope and its y-intercept. These geometric properties can often be interpreted in terms of the context. The SAT Math Test has questions that assess your ability to make these interpretations. For example, look back at the contexts in Examples 1 to 3. You created a linear function, $f(n) = 783 + 8n$, that describes the number of miles of paved road County X will have n years after 2014. This equation can be graphed in the coordinate plane, with n on the horizontal axis and $f(n)$ on the vertical axis. This graph is a line with slope 8 and vertical intercept 783. The slope, 8, gives the number of miles of new paved roads added each year, and the vertical intercept gives the number of miles of paved roads in 2014, the year that corresponds to $n = 0$.

EXAMPLE 14

A voter registration drive was held in Town Y. The number of voters, V, registered T days after the drive began can be estimated by the equation $V = 3{,}450 + 65T$. What is the best interpretation of the number 65 in this equation?

A) The number of registered voters at the beginning of the registration drive
B) The number of registered voters at the end of the registration drive
C) The total number of voters registered during the drive
D) The number of voters registered each day during the drive

The correct answer is choice D. For each day that passes, it is the next day of the registration drive, and so T increases by 1. When T increases by 1, the value of $V = 3{,}450 + 65T$ increases by 65. That is, the number of voters registered increased by 65 for each day of the drive. Therefore, 65 is the number of voters registered each day during the drive.

You should note that choice A describes the number 3,450, and the numbers described by choices B and C can be found only if you know how many days the registration drive lasted; this information is not given in the question.

Mastery of linear equations, systems of linear equations, and linear functions is built upon key skills such as analyzing rates and ratios. Several key skills are discussed in the next domain, Problem Solving and Data Analysis.